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A Reduced System of Inequalities for the Bargaining Sets $\mathcal{M}_1$, $\mathcal{M}_1^{(i)}$

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A REDUCED SYSTEM OF INEQUALITIES FOR THE BARGAINING SETS $\mathcal{M}_1, \mathcal{M}_1^{(i)}$

by

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C. 1

The basic ideas behind bargaining sets of cooperative games are given by R. J. Aumann and M. Maschler in [1]. The sets $\mathcal{M}_1^{(i)}$ are dealt with in [2] by M. Davis and M. Maschler and [4] by B. Peleg. In [3], Maschler presents a system of inequalities for $\mathcal{M}_1^{(i)}$.

This note presents a method for expressing $\mathcal{M}_1^{(i)}$ as a system of inequalities in a more compact form than that proposed in [3].

Let $N = \{1,\dots,n\}$ be the set of players of game G. Let $\mathcal{C} = \{B \mid B \subseteq N,\ B \text{ not empty}\}$ be the set of coalitions and let v be a real-valued function on $\mathcal{C}$. v is called the characteristic function of G, which can be written (N,v).

Let $x^N = (x_1,\dots,x_n)$ be a real-valued vector and define the excess of B for $B \in \mathcal{C}$. $e(B) = v(B) - \sum_{i \in B} x_i$.

Let a coalition structure, $\mathcal{B} = \{B_1,\dots,B_m\}$ be a set of disjointed coalitions whose union is N. x^N is a payoff vector for $\mathcal{B}$ if $e(B) = 0$ for all $B \in \mathcal{B}$.

Let $(x^N, \mathcal{B})$ be a payoff configuration, i.e. $\mathcal{B}$ is a c.s. and x^N is a p.v. for $\mathcal{B}$. Let i and j be two distinct players belonging to the same $B \in \mathcal{B}$. Suppose $C \in \mathcal{C}$ is such that $i \in C$, $j \notin C$ and $e(C) > 0$. Then we say i has an objection against j through C and represent the objection by $C, i \longrightarrow j$.

We say that j has a counterobjection to $C, i \longrightarrow j$ through $D \in \mathcal{C}$ if

(i) $j \in D$ and $i \notin D$, and

(ii) either $e(D) \geq e(C)$ or

both $D \cap C = \phi$ and $e(D) \geq 0$.

We represent this counterobjection by $C,i \longleftarrow j,D$.

The p.c. (x^N,B) is said to belong to the bargaining set $\mathcal{M}_1$ if for every objection $C,i \longrightarrow j$ there is a counterobjection $C,i \longleftarrow j,D$.

The following result presents $\mathcal{M}_1$ as a conjunction of a disjunction of a conjunction of linear inequalities. For simplicity we first assume the grand coalition c.s. $\mathcal{B} = \{N\}$. The general case is considered later.

THEOREM: $(x^N,\{N\}) \in \mathcal{M}_1 \iff$

$$\bigwedge_{C \in \mathcal{C}-\{N\}} e(C) \leq 0 \;\cdot\; \bigvee_{\tau \in \mu} \bigwedge_{D \in \tau, D \cap C \neq \phi} e(D) \geq e(C)$$

$$\bigwedge_{D \in \mathcal{C}, D \cap C = \phi} e(D) \geq 0$$

where μ is the set of minimal coverings τ of $N-C$ s.t. for each $j \in N-C$

$$\bigcup_{D/j \in D, D \in \tau} (C-D) = C \quad .$$

Proof: $(\Rightarrow)$. Suppose that $(x^N,\{N\}) \in \mathcal{M}_1$ and suppose that for some C, $e(C) > 0$. Then $C,i \longrightarrow j$ holds for all $i,j \in N$, $i \neq j$, with $i \in C$ and $j \in N-C$.

Fix $j \in N-C$. Then for each $i \in C$, there is a $D_i \in \mathcal{C}$ s.t. $C,i \leftarrow j,D_i$. Letting i vary over C we have

$$(*) \qquad \bigcup_i (C-D_i) = C \text{ , since } i \in C-D_i \quad .$$

Now for this j we can choose a minimal subcollection $\mathcal{K}_j \subseteq \{D_i\}$ satisfying (*). Let $\mathcal{K} = \bigcup_{j \in N-C} \mathcal{K}_j$. $\mathcal{K}$ is a covering of $N-C$ having the required property (*) for each j. Either $\mathcal{K}$ is a minimal covering satisfying (*) for each j or we can choose such a minimal subcollection $\mathcal{K}' \subseteq \mathcal{K}$. Every minimal covering can be reached in this way.

($\Leftarrow$). Suppose the right side of $\Leftrightarrow$ holds and suppose also that $(x^N,\{N\})$ does not belong to $\mathcal{M}_1$. Then there is a $C \in \mathcal{C}$ with a justified objection $C,i \rightarrow j$. That is, $e(C) > 0$, $j \notin C$ and $i \in C$ and for all $D \in C_j$

$$j \in D \Longrightarrow i \in D \vee (D \cap C \neq \phi \,\&\, e(D) < e(C))$$
$$\vee (D \cap C = \phi \,\&\, e(D) < 0) \quad .$$

But there is a $\mathcal{K} \in \mu$ and a $D \in \mathcal{K}$ such that $j \in D$ and

$$i \notin D \,\&\, D \cap C = \phi \,\&\, e(D) \geq 0$$

or

$$i \notin D \,\&\, D \cap C \neq \phi \,\&\, e(D) \geq e(C) \quad .$$

This is a contradiction completing the proof.

In the general case $\mathcal{B} = \{B_1,\ldots,B_m\} \neq \{N\}$ we let μ be the set of minimal coverings $\mathcal{K}$ of $\bigcup_k B_k - C$ over k such that

$C \cap B_k$ is non-empty with the following property:
For $j \in B_k - C$, $\{D/j \in D,\ D \in \mathcal{T}\}$ satisfies

$$\bigcup_{D} (C-D) \cap B_k = C \cap B_k \quad .$$

From the observation that every such covering $\mathcal{T}$ can be extended to a minimal covering in the case of the grand coalition we can conclude that the expression for $\mathcal{M}_1$ is at least as short in the general case as in the case of the grand coalition.

For the two person game, the solution is

$$e(1) \leq 0 \vee e(2) \geq 0$$

and

$$e(2) \leq 0 \vee e(1) \geq 0$$

which says that the excesses of the singleton coalitions are both non-negative or both non-positive.

For the three person game, the solution is (for the c.s. {123}, the grand coalition)

$$e1 \leq 0 \vee e\,23 \geq 0 \vee (e2 \geq 0 \text{ and } e3 \geq 0)$$

and

$$e2 \leq 0 \vee e\,13 \geq 0 \vee (e1 \geq 0 \text{ and } e3 \geq 0)$$

and

$$e3 \leq 0 \vee e\,12 \geq 0 \vee (e1 \geq 0 \text{ and } e2 \geq 0)$$

and

$$e12 \leq 0 \vee e3 \geq 0 \vee (e13 \geq e12 \,\&\, e23 \geq e12)$$

and

$$e23 \leq 0 \vee e1 \geq 0 \vee (e12 \geq e23 \,\&\, e13 \geq e23)$$

and

$$e12 \leq 0 \vee e2 \geq 0 \vee (e12 \geq e13 \,\&\, e23 \geq e13) \quad .$$

Under the assumption of "individual rationality" $ei \leq 0$ for each player i since he will not accept less than he can get alone. The bargaining set $\mathcal{M}_1^{(i)}$ thus consists, for the three person game, of the last three conjunctions, the first three being trivially true.

As a final example we use a 4-person game described in [1], Section 8, $v(1) = v(2) = v(3) = v(4) = 0$, $v(12) = a$, $v(23) = b$, $v(34) = c$, $v(13) = d$, $v(24) = e$, $v(14) = f$, $a,b,c,d,e,f \geq 0$. (We can consider v as extended to be zero on the three-person coalitions.) For this game the bargaining set $\mathcal{M}$ of [1] is the same as $\mathcal{M}_1^{(i)}$. We get the same solution given in the appendix of [1] for $B = \{12,34\}$

$$e12 = 0 \text{ and } e34 = 0 \text{ and}$$

$$e13 \leq 0 \vee e24 \geq 0 \vee (e2 = 0 \text{ and } e4 = 0) \vee (e2 = 0 \text{ and } e14 \geq e13 \text{ and } e34 \geq e13) \vee (e4 = 0 \text{ and } e12 \geq e13 \text{ and } e23 \geq e13) \vee (e14 \geq e13 \text{ and } e34 \geq e13 \text{ and } e12 \geq e13 \text{ and } e23 \geq e13)$$

and corresponding disjunctions for the coalitions 24,23,14. The three-person coalitions cannot provide objections since $v(C) = 0$, $v(\{i\}) = 0$ for $i \in C \Longrightarrow e(C) \leq 0$. This set of disjunctions is only slightly different in form (but equal in content) from that of [1]. However, no general method is presented there for determining such a system.

The author is grateful to Alan J. Hoffman for pointing out the simplification found possible in this note.

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